SuGra on G_2 Structure Backgrounds that Asymptote to AdS_4 and Holographic Duals of Confining $2+1d$ Gauge Theories with $\mathcal{N}=1$ SUSY

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Abstract :

In this work the solution generated by performing a U-duality on a deformation of the Maldacena-Nastase solution is studied. This is a solution of type-IIA with a metric that is asymptotically AdS_4 and supports a G_2 structure. It is believed to be dual to a $2+1d$, $\mathcal{N}=1$ gauge theory similar to the baryonic branch of Klebanov-Strassler with an additional intermediate scale. An improved radial coordinate is used allowing the derivation of UV series solutions to the BPS equation that persist to all orders. A study of the properties of the dual field theory is performed which includes an operator analysis, Wilson loops and a proposal for gauge couplings. The gauge theory dual appears to be a confining Chern-Simons quiver with gauge couplings that become constant at high energies.

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1. INTRODUCTION

In [1] Gaillard and Martelli introduce a solution generating technique which maps $\mathcal{N} = 1$ type IIA solutions supporting a G_2 structure into more general $\mathcal{N} = 1$ type IIA solutions with interpolating G_2 structure. This technique is one of a collection which are generally referred to as "Rotation", as this is how the procedure acts on the space of killing spinors (see [2, 3, 27] on $SU(3)$ structure rotations). The rotation of [1] has to be applied to a "seed" solution of type-II SuGra with an unwarped metric and NS 3-from flux only. After applying the rotation procedure you are mapped to a type-IIA solution with a warped metric, NS 3-from and RR 4-form flux which we refer to as the Gaillard-Martelli solution. In [1] the rotation is applied to a deformation, due to Canoura, Merlatti and Ramallo [6], of the Maldacena-Nastase solution [5]. They show that the rotation gives rise to a background with NS5 and D2 branes (it is argued in [7] that there are also D4-branes).

The Maldacena-Nastase solution is dual, in the IR, to the large N_c limit of $2+1$ dimensional $\mathcal{N} = 1$ SYM with Chern-Simons level $k = \frac{N_c}{2}$. It is generated by 5-branes which wrap a 3-cycle which grows form zero in the IR to infinity in the UV. Thus in the IR the gauge theory living on the world volume of the branes is 3 dimensional, but as we flow towards the UV the branes unwrap and the world volume gauge theory becomes 6 dimensional. From this it is clear that the Maldacena-Nastase solution is dual to a theory that is not UV complete. The Rotation procedure provides such a completion through extra warp factors introduced into the metric, these ensure that the 3-cycle on which the branes are wrapped remains finite in the UV.

It is important to appreciate the close connection between the Gaillard-Martelli solution and the baryonic branch, [9], of Klebanov-Strassler [8]. In [11] it was shown that starting from a deformation, [12], of the Maldacena-Nunez solution, [10], one can apply an $SU(3)$ rotation and generate the baryonic branch solution. It is U-duality which is discussed in [11] but this is equivalent to rotation, [2]. The Gaillard-Martelli solution is generated from the $2+1$ dimensional equivalent of the deformed Maldacena-Nunez solution and so we expect it to have similar properties. In particular we expect it to be dual to a cascading 2 node quiver which, being $2+1$ dimensional, will also have a Chern-Simon term. It is interesting to note that whilst the UV of the baryonic branch is Klebanov-Strassler, the UV of the Gaillard-Martelli solution is $AdS_4 \times Y$ where Y is the metric at the base of a G_2 cone.

In [2] a generalisation of the baryonic branch of Klebanov-Strassler was derived. This was achieved by applying the same $SU(3)$ rotation to the dual to $3+1$ dimensional $\mathcal{N} = 1$ SQCD with massless flavours [12]. It was observed that the numerology of the resulting solution was that of a modified two node quiver with both a duality and a higgsing cascade. There is no separate flavour symmetry (gauged or other wise) after rotation just a modification to the original gauge groups. However, this solution contains two pathologies, the first is the IR flavour singularity inherited from using [12] as a seed solution. The second is a rapidly increasing number of D3 branes in the UV that causes the metric to deviate from the desirable Klebanov-Strassler asymptotics. Both these issues are solved in [15], where a modified version of the dual of SQCD with massive flavours,[16], is used as a seed solution. Massive flavours are added in [16] by means of a flavour profile which interpolates between 0 in the IR, so there are no flavours and thus

no singularity, and 1 in the UV where the flavours are effectively massless. The idea of [15] was to use a profile which also dies off in the UV which allows the inclusion of an additional intermediate scale whilst maintaining the Klebanov-Strassler asymptotics.

With the insight gain from the aforementioned $SU(3)$ structure solutions the dual of $2+1$ dimensional SQCD with massive flavours was derived in [7]. This work also used this as a seed to generate a generalisation of the Gaillard-Martelli solution which is a G_2 structure equivalent of [15]. Here also it is necessary to introduce a profile that grows from the IR but dies away again in the UV to maintain the AdS asymptotics. As discussed in [1, 7], there are some technical difficulties in analysing the field theory dual to the Gaillard-Martelli solution and its generalisation. It is not clear how to make the same sort of matching of quiver numerology which is possible for Klebanov-Strassler. This is in part due to it being a running integral of $C_{(3)}$, rather than $B_{(2)}$, which must presumably be used to define the cascade. So it seems that we must use less direct methods to probe the dynamics of the dual gauge theory and this is the focus of this work.

The main part of the paper is divided into 2, with Section 2 concentrating on aspects of the SuGra solution and Section 3 focusing on aspects of the field theory dual.

In Section 2.1 the generalised Gaillard-Martelli solution is reviewed and an improved radial coordinate is introduced. The BPS equations are then solved in terms of this new radial coordinate in Section 2.2 where, unlike previous attempts, a UV series expansion is derived which persists to all orders. As with the deformed Maldacena-Nunez solution considered in [14] this expansion is in both exponentials and polynomials. The new radial variable also allow the more physically motivated profile of [15] to introduce the intermediate scale and numerical matching of the IR and UV asymptotic solutions is shown for this profile. In Section 2.3 we show that the metric is asymptotically the product AdS_4 of and the compact metric at the base of a G_2 cone and explicitly show the exact solution that can be extracted from a limit of parameter space. Section 2.4 then presents the Page and Maxwell charges for the various branes supported by the backgrounds of both the exact and more general solutions.

Attention is then turned to the field theory dual. In Section 3.1 an operator analysis is performed which is only possible because of the improved UV expansion. Confinement is then shown for the generalised system via a study of Wilson loops in 3 were the intermediate scale is also seen via a first order phase transition. Then in Section 3.3 a proposal is made for the gauge couplings that is consistent with a confining Chern-Simions theory in the IT.

The results are summarised in Section 4 were comments on future directions are also made. Finally there are two appendices, Appendix A presenting the BPS equations and Appendix B their general semi analytic UV solution in terms of 4 independent integration constants.

2. ON THE SUPERGRAVITY

2.1. The Type-IIA SuGra Set Up

In the purpose of this section is to briefly review the generalised Gaillard-Martelli solution more details can be found in [1, 7]. The string frame metric is given by:

$$ds^2_{\rm str} = N_c\left(\frac{1}{c\sqrt{H}}dx^2_{1,2} + \sqrt{H}ds^2_7\right) \tag{2.1}$$

where the internal part of the metric, ds^2_7, describes a manifold supporting a G_2 structure. This and the warp factor, H, are given by:

$$\begin{aligned} ds^2_7 =& e^{2h}dr^2 + \frac{e^{2g}}{4}(\sigma^i)^2 + \frac{e^{2g}}{4}(\omega^i - \frac{1}{2}(1+w)\sigma^i)^2 \\ H =& 1 - (\tanh\beta)^2 e^{2(\phi_\infty - \phi^{(0)})} \end{aligned} \tag{2.2}$$

Notice that we are using a different definition of holographic coordinate, r, to that previously used in [1, 7]. This will be the key to finding the improved asymptotic series. The functions g, h, w and ϕ_0 all depend on r only. The constant ϕ_∞ is the asymptotic value of $\phi^{(0)}$ in the U.V and β parametrises the interpolation of the G_2 structure. The constant c is a parameter which enters into the asymptotic UV solutions to the BPS equation 2.16. σ^i and ω^i are 2 $SU(2)$ left invariant 1-forms which satisfy the following differential relation:

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k; \qquad d\omega^i = -\frac{1}{2}\epsilon_{ijk}\omega^j \wedge \omega^k \tag{2.3}$$

These can be represented by introducing 3 angles for σ^i, $(\theta_1, \phi_1, \psi_1)$ and a further 3 for ω^i, $(\theta_2, \phi_2, \psi_2)$ such that:

$$\begin{aligned} \sigma^1 &= \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1 \\ \sigma^2 &= -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1 \\ \sigma^3 &= d\psi_1 + \cos\theta_1 d\phi_1 \end{aligned} \tag{2.4}$$

and similarly for ω^i. The angles are defined over the ranges: $0 \le \theta_{1,2} \le \pi$, $0 \le \phi_{1,2} < 2\pi$ and $0 \le \psi_{1,2} < 4\pi$

This type-IIA solution includes 2 non trivial fluxes, an RR 4-form $F_{(4)}$ and a NS 3-form $H_{(3)}$. They are given by:

$$\begin{aligned} F_{(4)} &= -N_c^{3/2}e^{-\phi_\infty}\coth(\beta)Vol_{(3)} \wedge dH^{-1} + N_c^{1/2}\sinh(\beta)e^{\phi_\infty - 2\phi^{(0)}} *_7 H_{(3)} \\ H_{(3)} &= \tfrac{N_c}{4}\Big[\left((\kappa + \tfrac{1}{2} + \tfrac{3x}{2}(C-1)P)\sigma^1\wedge\sigma^2\wedge\sigma^3 - \omega^1\wedge\omega^2\wedge\omega^3\right) + \\ & \tfrac{4xP'\eta + \gamma'}{2} dr \wedge \sigma^i \wedge \omega^i + \\ & \tfrac{1}{4}\epsilon_{ijk}\left((1+\gamma)\,\sigma^i\wedge\sigma^j\wedge\omega^k - (1+\gamma-2xP)\,\omega^i\wedge\omega^j\wedge\sigma^k\right)\Big] \end{aligned} \tag{2.5}$$

Where the function γ depends on r only, and η is given by Eq.A4.κ and C are constants that will be fixed below. The dilaton is given by:

$$e^{2\phi} = \cosh(\beta) e^{2\phi^{(0)}} H^{1/2} \tag{2.6}$$

The function P is a Profile which generalises the solution considered in [1]. It interpolates between 0 and 1 and introduces an intermediate scale into the theory that is parametrised by the constant x. From the smeared flavour brane origin of this solution (see [7]) it is natural to set $x = \frac{N_f}{N_c}$ but as we do not expect the dual field theory to have an explicit flavour symmetry[1] this point will not be laboured. A smeared brane configuration is introduced via a violation of the Bianchi identity for $H_{(3)}$:

$$\begin{aligned} dH_{(3)} &= \Xi_4 = \\ & -\tfrac{xN_c}{4}\Big[\tfrac{1}{4}\epsilon_{ilm}\epsilon_{ijk}\sigma^l \wedge \sigma^m \wedge \omega^j \wedge \omega^k + \eta P' \epsilon_{ijk} dr \wedge \sigma^i \wedge \omega^j \wedge \omega^k \\ & -\tfrac{3}{2}(C-1)P' dr \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \tfrac{1}{2}(2\eta+1)P'\epsilon_{ijk} dr \wedge \sigma^i \wedge \sigma^j \wedge \omega^k\Big] \end{aligned} \tag{2.7}$$

When $P = 0$ we have the solution considered in [1]. $P = 1$ is the solution generated by applying the rotation procedure to massless flavour solution derived in [6]. This has a singularity in the IR and a fast growing number of D2 branes in the UV which gives undesirable asymptotics [2]. In [7], these issues are resolved by a profile which kills off the $H_{(3)}$ source in both the far IR and UV.

From this point on we set:

$$C = 1; \qquad \kappa = \frac{1}{2} \tag{2.8}$$

C is set to this value to enable the dual gauge theory to have a quantised Chern-Simons term [7]. κ must be thus set to avoid a curvature singularity in the IR [5, 7].

We will be interested in the limit $\beta \to \infty$ which will require that the we make the identification:

$$c = e^{2\phi_\infty} \sinh\beta \tag{2.9}$$

As explained in [1] combination has to be held fixed if we want the limit $\beta \to \infty$ to be well defined[3]. It allows us to rewrite the dilaton and RR 4-form such that they remain finite when $\beta \to \infty$ as:

$$\begin{aligned} e^{2\phi} &= cH^{1/2} e^{2(\phi^{(0)} - \phi_\infty)} \\ F_{(4)} &= -\tfrac{1}{c^2} N_c^{3/2} Vol_{(3)} \wedge dH^{-1} + \tfrac{1}{c^{1/2}} N_c^{1/2} e^{2(\phi_\infty - \phi)} *_7 H_{(3)} \end{aligned} \tag{2.10}$$

[1] See also [15] in the context of the conifold.

[2] This was first observed in [2] for D3 branes on the conifold

[3] It should be noted however that in [1] it was a different constant that was held fixed. What was referred to as c there is g_0 here in 2.15

Specifically the independent components of $F_{(4)}$ are given by:

$$\begin{aligned}
F^{(4)}_{r123} &= -\frac{2}{\sqrt{cN_c H}} e^{-3g+2(\phi_\infty-\phi^{(0)})}; & F^{(4)}_{r\hat{i}\hat{j}k} &= \frac{1}{2\sqrt{cN_c H}} \epsilon_{ijk} \left(1+w^2-4xP-2w\gamma\right) e^{-g-2h+2(\phi_\infty-\phi^{(0)})}; \\
F^{(4)}_{r\hat{1}\hat{2}\hat{3}} &= -\frac{1}{\sqrt{4cN_c H}} V e^{-3h+2(\phi_\infty-\phi^{(0)})}; & F^{(4)}_{r\hat{i}jk} &= \frac{1}{\sqrt{cN_c H}} \epsilon_{ijk} \left(w-\gamma\right) e^{-2g-h+2(\phi_\infty-\phi^{(0)})}; \\
F^{(4)}_{txyr} &= -\frac{H'}{\sqrt{cN_c} H^{3/2}}; & F^{(4)}_{i\hat{i}j\hat{j}} &= \frac{1}{2\sqrt{cN_c H}} \left(4x\eta P' + \gamma'\right) e^{-2g-h+2(\phi_\infty-\phi^{(0)})};
\end{aligned} \tag{2.11}$$

Where V and η are defined in Eq.A4. The components are expressed in the following vielbein basis:

$$\begin{aligned}
e_R^{x^i} &= \sqrt{cN_c} H^{-1/4} dx^i, & e_R^r &= \sqrt{N_c} H^{1/4} e^g dr, \\
e_R^i &= \sqrt{N_c} H^{1/4} e^h \frac{\sigma^i}{2}, & e_R^{\hat{i}} &= \sqrt{N_c} H^{1/4} e^g \left(\frac{\omega^i - \frac{1}{2}(1+w)\sigma^i}{2}\right)
\end{aligned} \tag{2.12}$$

Finally, a potential $C_{(3)}$ such that $dC_{(3)} = F_{(4)}$ is given by:

$$\begin{aligned}
C^{(3)}_{123} &= \frac{e^{2(\phi_\infty-\phi^{(0)})}}{\sqrt{c}H^{3/4}} \cos\alpha; & C^{(3)}_{i\hat{j}\hat{k}} &= \epsilon_{i\hat{j}\hat{k}} \frac{e^{2(\phi_\infty-\phi^{(0)})}}{\sqrt{c}H^{3/4}} \cos\alpha \\
C^{(3)}_{\hat{1}\hat{2}\hat{3}} &= \frac{e^{2(\phi_\infty-\phi^{(0)})}}{\sqrt{c}H^{3/4}} \sin\alpha; & C^{(3)}_{\hat{i}j\hat{k}} &= -\epsilon_{\hat{i}j\hat{k}} \frac{e^{2(\phi_\infty-\phi^{(0)})}}{\sqrt{c}H^{3/4}} \sin\alpha \\
C^{(3)}_{txy} &= \frac{1}{\sqrt{c}H^{1/4}}; & C^{(3)}_{ri\hat{i}} &= \frac{e^{2(\phi_\infty-\phi)}}{\sqrt{c}H^{3/4}}
\end{aligned} \tag{2.13}$$

Where α is defined in Eq.A3.

2.2. Solutions to the BPS Equations

In this section we present improved (with respect to [1, 6, 7]) solutions to the BPS equations of Appendix A that give rise to asymptotically constant $\phi^{(0)}$.

It is only possible to solve the BPS equations of Appendix A analytically in terms of series expansions in the IR and the UV and then to show that these series can be smoothly connected over the full range of the holographic coordinate numerically. This was done for the profile dependent solution we consider here in [7], but in terms of a different radial coordinate[4] (see also [1, 6] for $P = 1$ and $P = 0$ cases respectively). All previous asymptotically constant $\phi^{(0)}$ solutions to the BPS system gave rise to series only valid up to a few orders in the UV expansion. This meant that one had to numerically shoot from the IR quite far into the UV to match the series expansions. The holographic coordinate, r, introduced here does not suffer from this pathology.

Although asymptotic solutions can be derived for quite arbitrary profile functions as in [7] the new radial variable allows the use of the physically well motivated profile of [15]:

$$P(r) = \theta(r - r_*) \tanh^4 2(r - r_*) e^{-4/3(r-r_*)} \tag{2.14}$$

[4] Specifically $\rho = e^{2h}$, were in Eq.2.1 $e^h dr_{\text{here}} = dr_{\text{there}}$

As with the solution to which this profile was originally applied, it is sufficient to ensure that the IR is singularity free and we keep the desired UV asymptotics. r_* is a constant which can be varied and defines where the new scale enters into theory. The transition between $r < r_*$ and $r > r*$ is continuous and so there is no singularity at $r = r_*$, this is because P, P', P'', P''' match as $r \to r_*+$ and $r \to r_*-$.

When $r_* = 0$ the new scale begins to appear at $r = 0$ and the IR expansion is given by:

$$\begin{aligned}
e^{2g} &= g_0 + \tfrac{(g_0-1)(9g_0+5)}{12g_0}r^2 + \left[\tfrac{(g_0-1)\left(54g_0^3+30g_0^2+25g_0+29\right)}{432g_0^3} - \tfrac{8\left(6g_0^2-3g_0+1\right)x}{3g_0^2}\right]r^4 + \\
&\quad \tfrac{8\left(203g_0^2-100g_0+41\right)x}{105g_0^2}r^5 + \ldots \\
e^{2h} &= g_0 r^2 - \left(\tfrac{3g_0^2-4g_0+4}{18g_0} + 32x\right)r^4 + 32xr^5 + \ldots \\
w &= 1 - \tfrac{3g_0-2}{3g_0}r^2 + \left[\tfrac{72g_0^3-84g_0^2-17g_0+38}{108g_0^3} + \tfrac{16(2g_0-1)x}{g_0^2}\right]r^4 - \tfrac{32(21g_0-10)x}{21g_0^2}r^5 + \ldots \\
\gamma &= 1 - \tfrac{r^2}{3} + \left[\tfrac{9g_0^2+4g_0-4}{108g_0^2} + \tfrac{16(3g_0-2)x}{3g_0}\right]r^4 - \tfrac{32(14g_0-11)x}{21g_0}r^5 + \ldots \\
\phi^{(0)} &= \phi_0 - \tfrac{7+576g_0x}{24g_0^2}r^2 + \tfrac{64x}{3g_0}r^3 + \left[\tfrac{\left(210g_0^2+56g_0-223\right)g_0^4}{1728} + \tfrac{2\left(71g_0^2-1\right)g_0^5x}{3}\right]r^4 + \ldots
\end{aligned} \tag{2.15}$$

The IR expansion when $r_* > 0$ is given by Eq.2.15 with $x = 0$. A series expansion of this type appears to persist to all orders in r.

The expansion in the UV takes the following form:

$$\begin{aligned}
e^{2g} &= ce^{4r/3} - 1 + \tfrac{33}{4c}e^{-4r/3} - \tfrac{3168-392xc}{72c^2}e^{-8r/3} + \\
&\quad \left[\tfrac{-840c^2c_\gamma-35840r+1860768}{7200c^3} + \tfrac{(13440rc-392808c)x}{7200c^3} + \tfrac{33x^2}{40c}\right]e^{-4r} + \ldots \\
e^{2h} &= \tfrac{3}{4}ce^{4r/3} + \tfrac{9}{4} - \left(\tfrac{77}{16c} - 3x\right)e^{-4r/3} + \tfrac{1536-88xc}{96c^2}e^{-8r/3} + \\
&\quad \left[\tfrac{360c^2c_\gamma+15360r-398912}{3200c^3} + \tfrac{x(168072c-5760rc)}{3200c^3} - \tfrac{1053x^2}{160c}\right]e^{-4r} + \ldots \\
w &= \tfrac{2}{c}e^{-4r/3} + \tfrac{22-6xc}{c^2}e^{-8r/3} + \tfrac{51-16xc}{2c^3}e^{-4r} + \ldots \\
\gamma &= \tfrac{1}{3} + xe^{-4r/3} + \left[\tfrac{3c^2c_\gamma+128r}{3c^2} - \tfrac{16rx}{c}\right]e^{-8r/3} + \\
&\quad \left[\tfrac{96-6c^2c_\gamma-256r}{3c^3} + \tfrac{(96rc-28c)}{3c^3}x\right]e^{-4r} + \ldots \\
\phi^{(0)} &= \phi_\infty + \tfrac{8+6xc}{4c^2}e^{-8r/3} - \tfrac{2(2+xc)}{c^3}e^{-4r} + \ldots
\end{aligned} \tag{2.16}$$

Unlike the $P = 0$ (or equivalently $x = 0$) solutions previously derived this series appears to persist to all orders in r. In fact there is a more general series for which this is true presented in Appendix B.

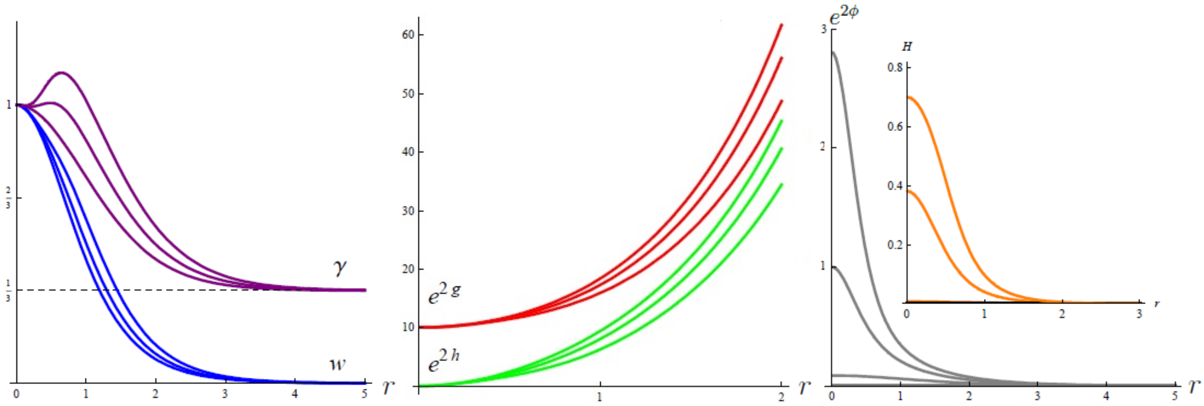

Figure 1: The graphs are numerical plots of the various functions of the solution for $g_0 = 10$, $x = 0, 1/2, 1$, $r_* = 0$. Increasing g_0 and x have different effects on each function and they have been grouped together by there behaviour. In the left panel the direction of increasing x is up. The IR and UV behaviours are independent of g_0. The width of the distribution of w solutions decreases as g_0 increases but γ does not change noticeably. Down is the direction of increasing x in the middle plot. The distributions of both e^{2g} and e^{2h} decrease as g_0 increases and the UV behaviour depends on g_0 as well. Increasing x is once more down in the right panel. As g_0 increases the IR value of H and $e^{2\phi}$ decreases as does the width of the distribution of each function. The UV values do not change.

Notice that like the deformed Maldacena-Nunez solution considered in [14] theses expansions are in both exponentials and polynomials. However the polynomial terms start at a more suppressed order. It is for this reason that the metric of this solution is asymptotically AdS rather than Log corrected AdS like (the Baryonic branch of) Klebanov-Strassler [8, 9], more on this in the next section.

There is in fact another UV expansion that solves the BPS equations. This gives rise to an asymptotically linear dilaton which, being unbounded, violates the reality of the metric[5] and so we do not consider it here (See [5–7] for details of solutions of this type)

Using the UV series expansions in Eqs.2.16 it is possible to derive the asymptotic behaviours of both the Dilaton, ϕ and the warp factor H which will be useful in the next section. They are given by:

$$\begin{aligned} e^{2\phi} &= \sqrt{3cx+4}e^{-4r/3} - \frac{2(cx+2)}{c\sqrt{3cx+4}}e^{-8r/3} + \frac{5696+9876cx+4467c^2x^2+216c^3x^3}{48c^2(3cx+4)^{3/2}} + \ldots \\ H &= \frac{3cx+4}{c^2}e^{-8r/3} - \frac{4(cx+2)}{c^3}e^{-4r} + \frac{3cx(91-120cx)+752}{24c^4}e^{-16r/3} + \ldots \end{aligned} \tag{2.17}$$

In Fig.1 there are plots of the numerical solutions of the BPS equations, Dilaton and H. These confirm that it is possible to smoothly connect the IR series expansions to the UV series expansion and also show the effect of varying x. It is worth making a comment

[5] Notice in Eq.2.2 that if the dilaton is unbounded then H can become negative. When this happens the $H^{1/2}$ factors in Eq.2.1 become imaginary

about how these solutions (and others) can be generated. A class of solutions can be defined by choosing values for x and r_* which parametrise the size of the intermediate scale and where it begins. The idea is to use Eq.2.15 to define IR boundary conditions for the BPS system at r_{min} close to zero then use a shooting technique to match the numerical solutions to Eq.2.16 for some range below a large but finite r_{max}. Given (x, r_*) there will be a minimum value $g_0 = g_{min}$ below which all the numerical solutions of to the BPS equations become badly singular and cannot be connected to a sensible UV expansion. When $g_0 = g_{min}$ we are led to asymptotically linear behaviour of $\phi^{(0)}$ mentioned above. For all $g_0 > g_{min}$ we are lead to a valid UV of the type in Eq.2.16 where c must be tuned to match each particular (g_0, x, r_*) combination. For all the solutions considered here it turned out that $c_\gamma = 0$. Thus give (x, r_*) there appears to be exactly one tunable parameter in each of the IR and UV and c increases non linearly with g_0.

2.3. AdS_4 Asymptotics and an Exact Solution for Infinite c

Using the UV series solutions of the previous section it is possible to show that the metric tends to the following form in the UV:

$$ds^2 = \frac{9\sqrt{4+3cx}N_c}{4}\left[\frac{4e^{4r/3}dx_{1,2}^2}{9(4+3cx)} + \frac{4}{9}dr^2 + \frac{1}{12}(\sigma^i)^2 + \frac{1}{9}(\omega^i - \sigma^i/2)^2\right] + O(e^{-4r/3}/c) \quad (2.18)$$

which is actually AdS_4 in disguise with an additional compact piece which describes the metric at the base of a G_2 cone. This can be elucidated with the following rescaling and coordinate transformations:

$$x^\mu \to \frac{3\sqrt{4+3cx}}{2}x^\mu; \qquad \rho = e^{2/3r} \quad (2.19)$$

In [1] the $P = 0$ case of this solution is studied with a different holographic coordinate. They show that when a certain parameter is taken to infinity an exact solution is can be extracted that is $AdS_4 \times Y$, where $Y \sim S^3 \times S^3$ which describes the UV of the Gaillard-Martelli solution. This solution is not conformal as the dilaton depends on their holographic coordinate r' as $\phi = -\frac{1}{2}\log(\frac{2r'}{9})$ which means it actually diverges in the IR. Its is possible to do something similar here and a virtue of the new radial coordinate is that the dilaton is IR finite. As discussed in [1, 7] there are some complications in interpreting the field theory dual to the Gaillard-Martelli solution and its generalisation and so it seems sensible to attempt to gain some insight from the taking a similar limit here.

The relevant limit is $c \to \infty$ as when this is taken the UV metric in Eq.2.20 becomes exact and finite. By construction the profile tends to zero in the UV and although x enters into the Eq.2.20 this is only as an overall factor which will not qualitatively change the physics. Thus it is $c \to \infty$ with $x = 0$ that will be explored. The metric of the exact solution is given by:

$$ds_{ex}^2 = \frac{9N_c}{2}\left[\frac{1}{9}dx_{1,2}^2 e^{4r/3} + \frac{4}{9}dr^2 + \frac{1}{12}(\sigma^i)^2 + \frac{1}{9}(\omega^i - \sigma^i/2)^2\right] \quad (2.20)$$

While the Dilaton, RR 4-form and NS 3-form are given by:

$$\begin{aligned}
\phi^{ex} &= \tfrac{1}{2}\Bigg[\log(2) - 4r/3\Bigg] \\
H^{ex}_{(3)} &= \tfrac{N_c}{4}\Bigg[\sigma^1\wedge\sigma^2\wedge\sigma^3 - \omega^1\wedge\omega^2\wedge\omega^3 + \tfrac{1}{4}\epsilon_{ijk}\left(\sigma^i\wedge\sigma^j\wedge\omega^k - \omega^i\wedge\omega^j\wedge\sigma^k\right)\Bigg] \\
F^{ex}_{(4)} &= -\tfrac{2N_c^{3/2}}{3}e^{8r/3}Vol_{(3)}\wedge dr - \tfrac{N_c^{3/2}e^{2r/3}}{4\sqrt{3}}dr\wedge\Bigg[\sigma^1\wedge\sigma^2\wedge\sigma^3 + \omega^1\wedge\omega^2\wedge\omega^3\Bigg]
\end{aligned} \tag{2.21}$$

A minimal choice for a potential such that $dC_{(3)} = F_{(4)}$ is given by:

$$C^{ex}_{(3)} = \frac{N_c^{3/2}e^{8r/3}}{4}Vol_{(3)} - \frac{\sqrt{3}N_c^{3/2}e^{2r/3}}{8}\Bigg[\sigma^1\wedge\sigma^2\wedge\sigma^3 + \omega^1\wedge\omega^2\wedge\omega^3\Bigg] \tag{2.22}$$

2.4. Page and Maxwell Charges

The purpose of this section is to learn something about the charges of the respective branes in this type-IIA solution. (see [18] for a discussion on Page and Maxwell charges) As pointed out in [7] we have an $F_{(4)}$ which is both a electric and magnetic brane source as well as $H_{(3)}$. So the branes in this solution are $NS5$, $D2$ and $D4$ branes. In [7] it was shown that the Maxwell-charge is running for the $D2$ and $D4$ branes and that it was only possible to define a distinct quantised page for the $D2$ branes. Thus we will attempt to see what can be learnt from the exact solution of Section 2.3 before proceeding any further.

It is clear from 2.21 that for the exact solution there is no cycle on which to define a D4 brane charge and so it must be zero. If we integrate the flux of $H^{ex}_{(3)}$ over the cycle $\omega^1\wedge\omega^2\wedge\omega^3$ we get:

$$-\frac{1}{4\pi^2}\int H^{ex}_{(3)} = N_c \tag{2.23}$$

which gives a quantised NS5 brane charge. The Maxwell charge of the D2 brane will clearly run as the pull back of $*F_{(4)}$ onto the only suitable cycle on which to define this charge is:

$$-\frac{\sqrt{3}}{16}N_c^{5/2}\sigma^1\wedge\sigma^2\wedge\sigma^3\wedge\omega^1\wedge\omega^2\wedge\omega^3 \tag{2.24}$$

However this is precisely equal to the pull back of $H^{ex}_{(3)}\wedge C^{ex}_{(3)}$ onto the same cycle[6]. So it is possible to define a page charge for the D2-branes:

$$M^{ex}_c = \frac{1}{(2\pi)^5}\int(*F^{ex}_{(4)} - H^{ex}_{(3)}\wedge C^{ex}_{(3)}) = 0 \tag{2.25}$$

[6] This was chosen rather than $B_{(2)}\wedge F_{(4)}$ for two reasons. As observed in [1] for the solutions considered here $C_{(3)}$ appears to pay something like the role $B_{(2)}$ does in Klebanov-Strassler. But also when we add the source for $H_{(3)}$, $B_{(2)}$ can no longer be defined.

But this is not really the whole story because $C^{ex}_{(3)}$ is not gauge invariant. So large gauge transformations can induce quantised shifts in M^{ex}_c. Consider such a large gauge transformation, $C^{ex}_{(3)} \to C^{ex}_{(3)} + \Delta C^{ex}_{(3)}$, where:

$$\Delta C^{ex}_{(3)} = -\frac{n\pi}{4}\left[\sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \omega^1 \wedge \omega^2 \wedge \omega^3\right] \tag{2.26}$$

Which is the 3-from equivalent of the gauge transformation considered in [19]. Under such a gauge transformation we have

$$M^{ex}_c \to M^{ex}_c + N_c \tag{2.27}$$

with N_c remaining invariant. This is rather reminiscent of the story in Klebanov-Strassler except $C_{(3)}$ is playing the role $B_{(2)}$ usually would. Indeed from Eq.2.21 it is clear that there is no cycle on which the flux of $B_{(2)}$ runs. However the flux of $C^{ex}_{(3)}$ as written in Eq.2.22 runs on the cycle $\sigma^i = \omega^i$:

$$c^{ex}_0 = -\frac{1}{4\pi^2}\int C^{ex}_{(3)} = \sqrt{3}N_c^{3/2}e^{2r/3} \tag{2.28}$$

And of course large gauge transformations will shift this as $c^{ex}_0 \to c^{ex}_0 + 2n\pi$.

When we consider the full solution of Section 2.1 the NS5 brane charge remains unchanged. More surprisingly we can also still define a D2 brane Page charge by integrating $*F_{(4)} - H_{(3)} \wedge C_{(3)}$ over the same cycle as Eq. 2.25. Further more, performing the same large gauge transformation on $C_{(3)}$, Eq. 2.26, shifts this page charge by the same quantised amount, $M_c \to M_c + nN_c$.

The flux of $C_{(3)}$ on $\sigma^i = \omega^i$ is no longer exact but is still running. Its asymptotic values are:

$$c_0 = -\frac{1}{4\pi^2}\int C_{(3)} = \begin{cases} N_c^{3/2}e^{2\Delta\phi}\left[\frac{g_0}{2\sqrt{c}}r^3 + \frac{4(4-3g_0)g_0-1}{24\sqrt{cg_0}}r^5 + ...\right] & r \approx 0 \\ \sqrt{3}N_c^{3/2}e^{2r/3}\left[1 + \frac{1}{2c}e^{-4r/3} + ...\right] & r \approx \infty \end{cases} \tag{2.29}$$

Where $\Delta\phi = \phi_\infty - \phi_0$. Similarly we can define a D2 brane Maxwell charge which runs on: $\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$:

$$Q_{D2} = \frac{1}{(2\pi)^5}\int *F_{(4)} = \begin{cases} \frac{N_c^{5/2}e^{2\Delta\phi}}{\sqrt{c}\pi}\left[\frac{\sqrt{g_0}(576g_0x+7)}{48}r^4 - 16g_0^{3/2}xr^5 + ...\right] & r \approx 0 \\ \frac{\sqrt{3}N_c^{5/2}}{\pi}\left[\frac{3cx+4}{8}e^{2r/3} + \frac{9cx+4}{16c}e^{-2r/3} + ...\right] & r \approx \infty \end{cases} \tag{2.30}$$

Numerical plots of c_0 and Q_{D2} can be seen in Fig.2. Note that since these quantities contain the UV parameter c, this must be matched to (g_0, x, r_*) for each specific solution as described towards the end of Section 2.2.

For the general solution it is also possible to define a running D4 brane Maxwell charge:

$$Q_{D4} = \frac{1}{(2\pi)^3}\int F_{(4)} = \begin{cases} \frac{N_c^{3/2}e^{2\Delta\phi}}{\sqrt{c}\pi}\left[\frac{\sqrt{g_0}}{24}r^2 - \frac{(g_0-1)(21g_0+25)}{864g_0^{3/2}}r^4 ...\right] & r \approx 0 \\ \frac{N_c^{3/2}}{\pi}\left[\frac{8cx-16+c^2c_\gamma+\frac{8}{3}r(16-6cx)}{4\sqrt{3}c^2}e^{-2r} + ...\right] & r \approx \infty \end{cases} \tag{2.31}$$

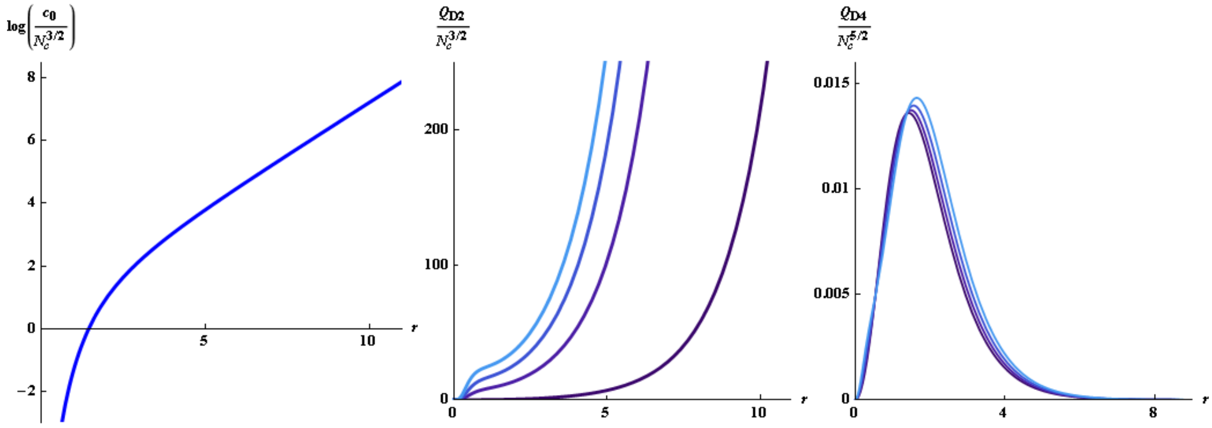

Figure 2: Plots of SuGra observables. The left panel is plots of $Log(c_0)$ for $x = 0$, $r_* = 0$ with $g_0 = 30$. The Middle and right panels contains plots of the Maxwell charges Q_{D2} and Q_{D4} for $g_0 = 40$, $r_* = 0$ and $x = 0, 1, 2, 3$ colour coded purple to blue.

where this quantity has been pulled back onto the 4-cycle defined by fixing ψ_1 and ψ_2 such that $\psi_1 - \psi_2 = (2k+1)\pi$ for integer k. These angles refer to the representation of the left invariant 1-forms in Eq.2.4.

The water muddies at this point. There is no suitable 4-form combination with which to define a quantised page charge for the D4 brane. However The D4 page charge dies off very quickly in the UV as $Q_{D4} \sim e^{-2r}$ compared to $c_0 \sim Q_{D2} \sim e^{2r/3}$. In fact numerical plot of Q_{D4} in Fig.2 shows that it is always small compared to these quantities and after first reaching a maximum close to the IR it once more dies off again very quickly.

3. ON THE FIELD THEORY

In section 2 a new radial variable was introduced and used to solve the BPS system of Appendix A. We do not believe that this represents a different SuGra solution to that presented in [7] but rather gives the best coordinate system to describe the solution. In particular for the first time it was possible to write a UV expansion for the (generalised) deformed Maldacena-Nastase solution that persists to all orders in r. Some aspects of the SuGra solution were also explored.

Using what has been learnt in the previous sections the objective from here on is to learn some new information about the dual field theory.

3.1. Operator Analysis

It was shown in Section 2.3 that the metric in the UV is asymptotically $AdS_4 \times Y$, where Y is a compact space. However we cannot have a CFT living on the boundary of the space as the dilaton tends to $\phi \sim -4r/3$ rather than a constant. None the less there is some hope that some insight my be gleamed from the AdS-CFT dictionary.

For a full study of the operators in the dual field theory one should integrate the

type-IIA action over the compact part of the background then perform a holographic renormalisation group analysis on the resulting 4-d theory in the spirit of [21]. However the literature on this subject deals with duals to 3+1d theories and deriving equivalent results for 2+1d theories is outside the scope of this work.

We look to [20] which presents a more ad-hoc method for extracting information about the operator content of a field theory from its dual gravity description, as long as the metric is asymptotically AdS.

When ρ is the standard AdS radius, ie $\rho = e^{2r/3}$, an asymptotic solution which behaves like:

$$a_i \rho^{\Delta-3} + b_i \rho^{-\Delta} \tag{3.1}$$

corresponds to an operator insertion into the Lagrangian of the field theory of the form:

$$\mathcal{L}' = \mathcal{L} + a_i \mathcal{O}_i \tag{3.2}$$

A non zero b_i then implies that this operator is picking up a vev:

$$< 0|\mathcal{O}_i|0 >= b_i \tag{3.3}$$

The case $a_i = 0$ with $b_i \neq 0$ can also be interpreted as a condensate.

Thus it sub leading terms in the UV expansions of the various terms that make up the metric that this analysis will be performed on. That is the leading order non AdS_4 deformations. The relevant combinations, written in terms of $\rho = e^{2r/3}$ are:

$$\begin{aligned}
\frac{1}{\sqrt{H}} &= \frac{\rho^2 N_c}{\sqrt{3cx+4}} + \frac{2N_c(cx+2)}{c(3cx+4)^{3/2}} + \frac{N_c(9cx(cx(120cx+101)-244)-1856)}{48c^2\rho^2(3cx+4)^{5/2}} + O\left(\frac{1}{\rho}\right)^4 \\
\sqrt{H}e^{2g}\frac{dr}{d\rho}^2 &= \frac{9N_c\sqrt{3cx+4}}{4\rho^2} - \frac{9(N_c(5cx+8))}{4\rho^4\left(c\sqrt{3cx+4}\right)} + O\left(\frac{1}{\rho}\right)^5 \\
\frac{\sqrt{H}e^{2g}}{4} &= \frac{3}{16}N_c\sqrt{3cx+4} + \frac{3N_c(7cx+8)}{16c\rho^2\sqrt{3cx+4}} + O\left(\frac{1}{\rho}\right)^4 \\
\frac{\sqrt{H}e^{2h}}{4} &= \frac{1}{4}N_c\sqrt{3cx+4} - \frac{N_c(5cx+8)}{4\rho^2\left(c\sqrt{3cx+4}\right)} + O\left(\frac{1}{\rho}\right)^4
\end{aligned} \tag{3.4}$$

these are the factors of $dx_{1,2}^2$, $d\rho^2$, and the two 3-spheres in the metric respectively. The leading terms of the first and second objects equations are the AdS terms so these will be ignored.

It is possible to see two different behaviours in Eq.3.4. The sub leading term of the first equation signals a $\Delta = 3$ operator insertion in the Lagrangian of the theory. This operator is marginal and does not have a vev in the solutions we consider in the bulk of this work. However ρ^{-3} terms that would give a vev to this operator appear in the general UV expansions of Appendix B. We see this behaviour once more in the leading terms of the last 2 equations. It is interesting to note that this behaviour is also seen in the UV expansion of γ (see Eq.2.16 and Eq.B1). We also see a sign of a $\Delta = 4$ vev in the sub leading factor of $d\rho^2$, it possible that this controls the IR dynamics of the theory.

3.2. Wilson Loops

In this section we will calculate the inter-quark potential between two massive non dynamical quarks. This can be extracted from the expectation value of a rectangular Wilson loop which extends in time, T, and space, L in the $T \to \infty$ limit. The spacial extent of the loop defines the separation of a quark anti-quark pair and the potential, $E(L)$, can be found via the identification:

$$< \mathcal{W}_{\Box} > \sim e^{-TE(L)} \tag{3.5}$$

Wilson loops are an effective tool in probing the behaviour of a gauge theory. In particular Wilson loops will obey an area law in the IR if the theory exhibits confinement, where by it is only the area rather than the precise shape of the loop that will determine its expectation value. If a theory is conformal the expectation value of a loop has to have a particular form determined by conformal invariance. In terms of the inter quark potential this amounts to the following behaviours:

- Confining behaviour: $E(L) \sim L$
- Conformal behaviour: $E(L) \sim \frac{1}{L}$

In the gauge-gravity correspondence, the gravitational dual of a Wilson loop is a minimal surface which extends from a D-brane in the UV down into the IR [22]. The UV boundary of this surface is given by the shape of the loop in the field theory with larger loops corresponding to surfaces that extend further along the holographic coordinate, and thus probe deeper into the IR. When the minimal surface has sufficient symmetry to be effectively 2-d we need only consider the Nambu-Goto action of a fundamental string so that:

$$E(L) = \frac{1}{T}\mathcal{S}_{\text{N.G}} \tag{3.6}$$

This string will fill be fixed in the UV at $r = \infty$ and extend down to some finite $r_{min} \geq 0$. The string also has to satisfy boundary conditions in the UV such that the coordinates that are parallel, $x_{||}$, and perpendicular, $x_\perp$, to the brane satisfy $\frac{dx_{||}}{dx_\perp} = 0$.

The method of calculating a rectangular Wilson loop via the Nambu-Goto action is well known for arbitrary metrics. [23] presents a rigorous derivation and contains extensive discussion that will not be repeated here. In order to get the correct normalisation factor for the Wilson loops the field theory coords must be rescaled such that they no longer have N_c as a factor. The relevant functions are then:

$$f(\rho)^2 = g_{xx}g_{tt} = \frac{1}{c^2H}; \quad g(\rho)^2 = g_{xx}g_{\rho\rho} = \frac{N_c e^{2g}}{c}; \quad V = \frac{f(\rho)\sqrt{f(\rho)^2 - f(\rho_{min})^2}}{f(\rho_{min})g(\rho)} \tag{3.7}$$

and the length and potential of the rectangular Wilson loop are given by:

$$\begin{aligned} L =& 2\int_{\rho_{min}}^{\infty} \frac{d\rho}{V}; \\ E =& f(\rho_{min})L + 2\int_{\rho_{min}}^{\infty} \frac{g(\rho)}{f(\rho)}(\sqrt{f(\rho)^2 - f(\rho_{min})^2} - f(\rho))d\rho - 2\int_0^{\rho_{min}} g(\rho)d\rho \end{aligned} \tag{3.8}$$

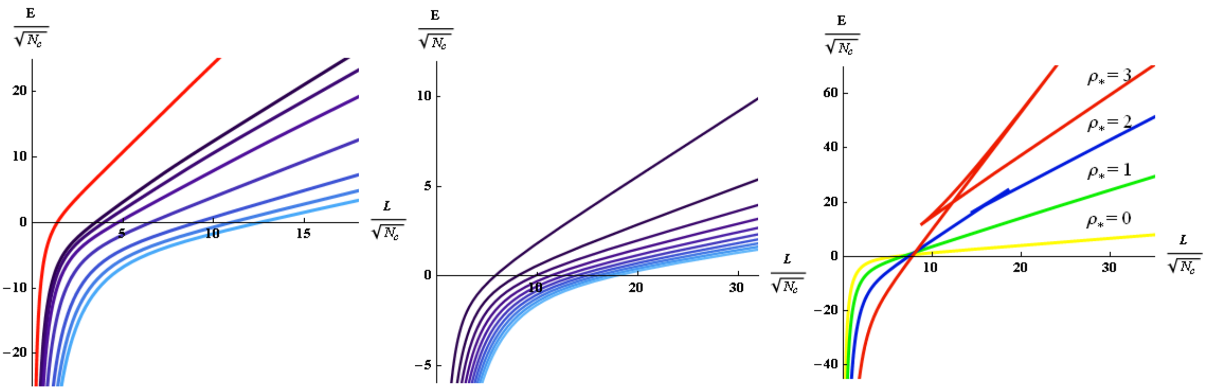

Figure 3: Plots of potential verses Length for rectangular Wilson loops. The left panel shows plots for $g_0 = 10$, $r_* = 1$ and $x = 1/4, 1/3, 1/2, 1, 2, 3, 4$ colour coded purple to blue, $x = 0$ is also shown in red. The middle panel has plots for $r_* = 0$, $x = 1$ and g_0 from 10 to 100 coded purple to blue once more. The right panel contains plots for $g_0 = 20$, $x = 1/2$ for increasing ρ_*

Where the expression for the potential has had the infinite quark mass's subtracted and been expressed in a way such that each term in the sum is finite. Notice that the UV parameter c appears in Eq. 3.7. This must be numerically determined for every combination of (g_0, x, r_*) under consideration as describe towards the end of Section 2.2. The strings boundary conditions at infinity amount to $V \to \infty$ as $\rho \to \infty$ which is easily shown to be satisfied (This would not be the case for finite β).

In Fig.3 there are numerical plots of the inter-quark potential, $E(L)$, for various values of (g_0, x, r_*). A detailed numerical study suggests that for all finite values of these parameters we are led to inverse power law behaviour in the UV and Linear behaviour in the IR which is consistent with the dual QFT exhibiting confinement. Further more, given (g_0, x) there exists $r_* = r_{crit}$ such that when $r_* < r_{crit}$ the transition between UV and IR behaviours is completely smooth. However the existence of the intermediate scale can be made manifest when $r_* > r_{crit}$. Then a first order phase transition appears just like for the Gibbs free energy vs. pressure curve of the Van der Waals gas. This behaviour was first observed in [24] (see also [25] for wilson loop calculations in a similar set up).

Given (x, r_*) the effect of increasing g_0 is more subtle. When $x = 0$ there is little effect at all, however for $x > 0$ increasing g_0 decreases the gradient of the liner behaviour in the IR, a sign that the QCD-like string tension is decreasing in the dual QFT. Increasing x for fixed (g_0, r_*) also decreases the gradient of the potential in the IR and so the QCD-like sting tension reduces also.

We can gain further insight into the asymptotic behaviours of the rectangular wilson loop using the expansions of section 2.2. The upper limit of the integral that defines L in Eq.3.8 is finite ($L \sim \int^{\infty} e^{-2r} dr$) while the lower limit is given by:

$$L \sim \int_{\rho_{min} \sim 0} (8\sqrt{3N_c} g_0 \sqrt{\frac{c g_0}{576 g_0 x + 7}} \sinh(\Delta\phi) \frac{1}{r} + ...) \sim \log(r) \tag{3.9}$$

where we have have introduced $\Delta\phi = \phi_\infty - \phi_0$. Eq.3.9 diverges for small r which indicates the absence of screening.

In [23] an exact expression for the rate of change of E with respect to L is derived:

$$\frac{dE}{dL} = f(\rho_{min}) \tag{3.10}$$

This equation can be used to derive an exact expression for the inter quark potential in terms of an expansion in large L, provided $r_{min}(L)$ can be found. In this case it can, we need only invert Eq.3.9 and integrate Eq.3.10 to arrive at:

$$E = \frac{1}{c\sqrt{1 - e^{2\Delta\phi}}} L + ... \tag{3.11}$$

The next term is a complicated power of e^{-L} that we will not quote explicitly. So when L becomes large and so the deep IR of the field theory is being probed this in a good approximation. Thus, in Eq.3.11, we explicitly see that in the IR the potential is linear and so consistent with confinement with QCD-string tension, σ, given by:

$$\sigma = \frac{1}{c\sqrt{1 - e^{2\Delta\phi}}} \tag{3.12}$$

At this stage we might want to ask about the $c \to \infty$ limit. We then have the exact solution described in Section 2.3 and the relevant part of the metric is AdS_4:

$$f^2 = \left(\frac{1}{2}\right)^2 \rho^4; \qquad g^2 = N_c \left(\frac{9}{2}\right)^2 \tag{3.13}$$

From this we can solve exactly for E, as in [22], and arrive at:

$$E = -(2\pi)^3 \frac{81\sqrt{N_c}}{2\Gamma(\frac{1}{4})^2 L} \tag{3.14}$$

Where $\rho = e^{2r/3}$.

This result, although preordained by the virtues of AdS, is curious. It is a sign of conformal invariance in a theory that we know cannot be, because of dilaton scales as $\phi \sim -2r/3$. Of course it should be noted that the metric of the theory is only asymptotically AdS_4 in string frame which is what the probe strings considered here see.

3.3. Gauge Couplings

The generalised Gaillard-Martelli solution presented in Section 2 have some sticking similarities to the (Baryonic branch [9]) of Klebanov-Strassler [8] and its generalisation [15]. There is a running $C_{(3)}$ form flux at infinity reminiscent of the running $B_{(2)}$ of Klebanov-Strassler and there are many types of branes contained in the set up, albeit in type-IIA rather than type-IIB. For this reason one might anticipate that the field theory dual will be a confining 2-node quiver which, being 2+1d will also have a Chern-Simons term like the Maldacena-Nastase solution [5]. If this is true it should be possible to define 2 gauge couplings and this section is towards that aim.

Consider a probe D2 brane which extends along the field theory directions with world volume flux F. If the DBI action of such a brane is expanded to order F^2, the coefficient at that order defines a gauge coupling:

$$\frac{1}{g_1^2} = e^{-\phi}\sqrt{-Det(\hat{G}_3)}H = e^{\phi_\infty - \phi^{(0)}} \tag{3.15}$$

Where $\hat{G}_3$ is the pull back of the metric onto (t, x, y), the factor H comes from $F_{\mu\nu}F^{\mu\nu} = H\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$ and we ignore constants factors. Clearly $\frac{1}{g_1^2} \sim 1$ in the UV which is a sign of dilation invariance in this coupling, it cannot be full conformal invariance as the dilaton depends on r. Curiously the coupling is also constant in the IR, $\frac{1}{g_1^2} \sim e^{\Delta\phi}$, and the coupling smoothly interpolates between these values as shown in Fig.4.

Another coupling may be defined via a probe D4-brane extended along the field theory directions and wrapping the 2-cycle defined by $\theta_1 = \theta_2$, $\phi_1 = \phi_2$, $\psi_1 = \psi_2$ =constant. The coupling is given by:

$$\frac{1}{g_2^2} = e^{-\phi}\sqrt{-Det(\hat{G}_5)}H = \sqrt{H}\left(4e^{2h} + e^{2g}(1-w)^2\right)e^{\phi_\infty - \phi^{(0)}} \tag{3.16}$$

where constant factors are once more ignored and $\hat{G}_5$ is induced metric. This coupling is also constant in the UV, $\frac{1}{g_2^2} \sim 8\sqrt{4 + 3cx}$, however in the IR the behaviour is strongly coupled, $\frac{1}{g_2^2} \sim r^2$. See Fig.4 for a numerical plot.

These couplings are sufficient for a 2-node quiver, however there is one further possibility, albeit less likely. The Gaillard-Martelli solution also contains fractional D2-branes which come from NS5-branes wrapping the cycle $\sigma^i = \omega^i$ [7]. It may be possible to define another coupling via a probe D2 instanton that wraps the 3-cycle on which the NS5-branes are wrapped[7]. If the definition is valid the coupling is given by:

$$\frac{1}{g_3^2} = e^{-\phi}\sqrt{-Det(\hat{G'}_3)} = \frac{\sqrt{H}e^{\phi_\infty - \phi^{(0)}}}{\sqrt{c}}\left(4e^{2h} + e^{2g}(1-w)^2\right)^{3/2} \tag{3.17}$$

where $\hat{G'}_3$ is the induced metric on $\sigma^i = \omega^i$. The advantage of this tentative coupling is that it is asymptotically free $\frac{1}{g_3^2} \sim e^{2r/3}$ as well as being strongly coupled in the IR where $\frac{1}{g_3^2} \sim r^3$. This coupling is numerically plotted in Fig.4.

It seems most likely that only g_1 and g_2 are good definitions of couplings and although g_3 cannot be conclusively excluded this will be assumed. As shown in Fig.4 g_1 and g_2 both asymptote to constants in the UV. This is inherited from the conformal symmetry of AdS_4, and while the symmetry is broken by the dilaton, this is evidence that the dual gauge theory has some residual dilation invariance at high energies.

In the IR the fact that g_1 starts to increase then becomes constant may appear strange, particularly in the light of the confining Wilson loops of Section 3. However one must appreciate that in YM-CS like theory's the coupling of the Maxwell term can increase as

[7] A D1 instanton is used to define the coupling of $\mathcal{N} = 1$ SQCD in [12]. This is a type-IIB dual with a geometry generated from D5-branes which wrap a 2-cycle. However it is not clear weather the same could work for a NS5 brane on a 3-cycle

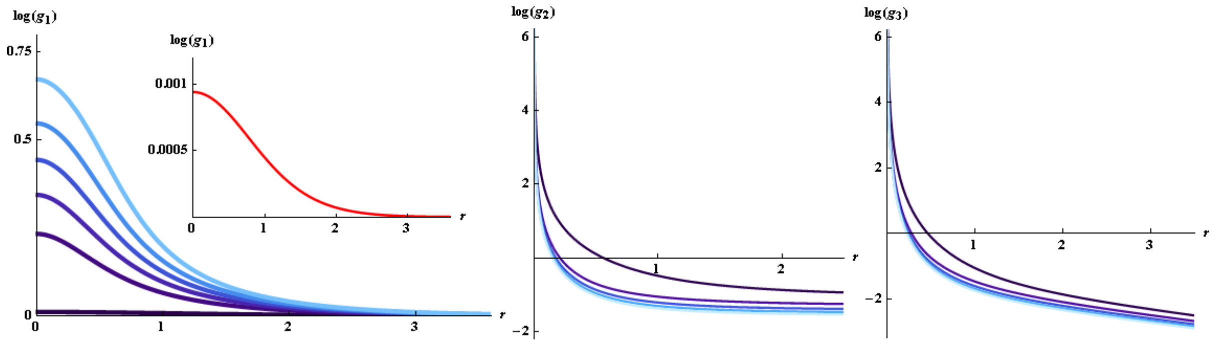

Figure 4: Log plots of 3 putative gauge couplings. The left panel shows $Log(g_1)$ for $g_0 = 10$, $r_* = 0$ with $x = 0, 1, 2, 3, 4, 5$ color coded purple to blue. As $x = 0$ is very small it has been plotted again on a different scale in red. The centre and right panels shows $Log(g_2)$ and $Log(g_3)$ respectively for $g_0 = 20$, $r_* = 0$ with x increasing from 0 to 1/2 in increments of 1/8.

one flows toward the IR and then become frozen in the strong coupling regime. Here the dynamics will become dominated by a, possibly confining, pure Chern-Simons theory. This mechanism is governed by the effective mass that the Chern-Simons level gives the gauge field, $g_{YM}^2|k|$. It is gratifying to see that the interpolating region of g_1 is of approximately the same width as that of γ and w in Fig.1, it is likely one of these functions is dual to an object in the gauge theory that governs this scale.

The IR behaviour of g_2 is exactly what one would expect from a theory exhibiting confinement. So it seems that, in a sense, g_1 is a Maxwell type coupling, while g_2 is a coupling which describes the dynamics on the full theory. However if g_1 and g_2 are indeed good definitions of gauge theory couplings for a 2-node quiver with gauge group $SU(N_a)\times SU(N_b)$, it is entirely possible that $\frac{1}{g_a^2}$ and $\frac{1}{g_b^2}$ are given by the sum and difference of $\frac{1}{g_1}$ and $\frac{1}{g_2}$ as with Klebanov-Strassler.

4. DISCUSSION

In this work a generalisation of the Gaillard-Martelli solution, [1] with an additional intermediate scale was considered which is a solution to type-IIA SuGra. This was first derived in [7] by applying an G_2 structure rotation (equivalently U-duality) to a generalisation of the deformed Maldacena-Nastase solution [5, 6] which includes a 5 brane profile.

An new radial coordinate was introduced that gives a UV series solution to the BPS equations of (generalised) deformed Maldacena-Nastase that persists to all order for the first time. This was shown to numerically match IR expansion and it was shown that (at least for all cases considered) there was only 1 tunable constant in both the IR and the UV. This puts the G_2 cone solutions on a more equal footing to with their Conifold cousins however equality would require a partial integration which is still lacking. The metric of the solution is asymptotically $AdS_4 \times Y$, where Y is the compact metric at the base of a G_2 cone. Conformal symmetry is broken however as the dilaton depends

on the radial coordinate as $\phi \sim -r$. In the limit where the UV parameter $c \to \infty$ the semi analytic UV series solutions become exact and the metric is precisely $AdS_4 \times Y$ with dilaton $\phi = 1/2(\log 2 - 4r/3)$, which unlike the result of [1] is IR finite. The exact solution has Maxwell charges for D2 and NS5 branes with the D2 charge running such that it is proportional to the flux $C_{(3)}$ on a certain cycle. A page charge can be defined for the D2 brane which is quantised and can only be defined up to shifts induced by large gauge transformations on $C_{(3)}$. The story is similar for the full generalisation of the Gaillard-Martelli solution however a Maxwell charge for D4 branes can now be defined in addition to D2 and NS5 branes. The Maxwell charge for D4 branes runs in such a way that it grows from the IR but then dies off again before the UV. Thus the D4 branes appear to be localised near the IR. It is interesting to note that the size of the cycle $\sigma^i = \omega^i$ exhibits a similar distribution. This is the cycle on which the NS5 branes of the deformed Maldacena-Nastase solution are wrapped [6, 7].

It is likely that the Gaillard-Martelli solution and its generalisation are dual to a 2 node quiver with gauge group $SU(N) \times SU(M)$. This is because of the similarity it bares to the baryonic baryonic branch of Klebanov-Strassler [8, 9]. Klebanov-Strassler also exhibits a duality cascade which could be mediated in the Gaillard-Martelli solution via the shifts in the D2 page charge under large gauge transformations, as in [19]. Proving this however is outside the scope of this work.

An operator analysis was performed which suggests that the Lagrangian of the dual field theory contains a dimension 3 operator insertion. It also indicated that there was a dimension 4 vev, which may be partially responsible for the IR dynamics.

It was shown that the dual gauge theory is confining via a study of rectangular Wilson loops. Unfortunately it is not clear how to back this result up with a calculation of the k-string tension in the spirit of [29]. This is because in the IR it is possible to perform a gauge transformation such that the compact part of the metric is a round S^3 and we require an S^3 inside an S^4 to follow the prescription of [29]. However the Wilson loop study does make the addition of the new scale clear with the appearance of a first order phase transition in a certain region of parameter-space. The recent results of [25] indicate that it should be possible to add further intermediate scales by modifying the NS5-brane profile. This may be of some use in holographic condensed matter model building.

Further evidence of confinement was shown by the gauge coupling. This behaviour was interpreted as coming from a pure Chern-Simons theory which dominates the IR, the evidence for which is the freezing of a Maxwell like coupling there. Both the well motivated couplings defined are constant in the UV which signals some kind of residual dilation symmetry at high energies. That despite the fact conformal invariance is broken by a non constant dilaton. The fact that there are two also fits in nicely with the picture of a 2-node quiver. However caution should be taken as it was not possible to completely disregard a potential coupling coming from wrapped NS5 branes. This unlikely coupling is asymptotically free.

This work and [7] show that it is possible to derive solutions to the BPS equations of deformed Maldacena-Nastase in terms of 2 different radial variables. It would be desirable to find the asymptotically linear dilaton solutions of [7] in the radial variable used here. For completeness the conifold solutions of [14] should also be found in a radial variable of the type used in [6, 7] ie $\rho \sim e^{2h}$.

An interesting future direction would be to try to find new IR solutions that can be

numerically matched to the general UV expansions in the appendix. These solutions would still have metrics that are asymptotically AdS_4 for arbitrary values of (c_h, c_w, c_γ). The question is which combinations of these values lead to a regular IR. Experience of similar Conifold solutions [26, 26] indicates that it should be possible find G_2 solutions which exhibit a similar walking type behaviour. A brief numerical investigation solving form the UV seems to suggest that this is so but a more detailed study is required and the IR must actually be derived.

It would also be of some interest to calculate the scalar (glue ball) spectrum of this theory as this is one of the few examples of solutions that are both confining and asymptotically AdS, albeit only in string frame and $2+1d$. An algorithm for doing this is presented in [28] and references therein which should be applicable, modulo small modifications accounting for the fact that the field theory dual here is not $3+1d$.

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Appendix A: The BPS equations

Here we quote the BPS equations for the definitions of the fields we use here. These results were derived for the flavour profile dependent system in [7] and are generalisations of the BPS systems derived in [1, 5, 6].

The $\beta \to 0$ limit of the system we describe in section2.1 gives the metric:

$$ds^2_{\rm str} = N_c\left(\frac{1}{c}dx^2_{1,2} + e^{2g}dr^2 + \frac{e^{2h}}{4}(\sigma^i)^2 + \frac{e^{2g}}{4}(\omega^i - \frac{1}{2}(1+w)\sigma^i)^2\right) \tag{A1}$$

with $H_{(3)}$ defined as in Eq.2.5 and $F_{(4)} = 0$. The conditions that $\mathcal{N}=1$ supersymmetry is satisfied in this limit imply the following differential equations for the various functions

in the metric and $H_{(3)}$:

$$
\begin{aligned}
\phi'^{(0)} &= \tfrac{1}{8}\Big[-Ve^{g-3h}\sin(\alpha) + 12e^{-g-h}(\gamma - w)\sin(\alpha) + 8e^{-2g}\cos(\alpha) + \\
&\qquad 6e^{-2h}\cos(\alpha)\left(4Px - w^2 + 2w\gamma - 1\right)\Big] \\
h' &= \tfrac{1}{8}e^{-g-3h}\Big[-4e^{g+h}\cos(\alpha)\left(e^{2g}\left(w^2-1\right) + 4Px - w^2 + 2w\gamma - 1\right) - \\
&\qquad 4e^{2h}\sin(\alpha)\left(\left(2e^{2g}-1\right)w + \gamma\right) + e^{2g}V\sin(\alpha)\Big] \\
g' &= \tfrac{1}{4}e^{-2h}\cos(\alpha)\left(e^{2g}\left(w^2-1\right) - 4Px + w^2 - 2w\gamma + 1\right) + \\
&\qquad e^{-g-h}(w-\gamma)\sin(\alpha) + \left(1 - e^{-2g}\right)\cos(\alpha) \\
w' &= \tfrac{1}{4}\Big[\left(2e^{-g-h}\sin(\alpha)\left(3e^{2g}\left(w^2-1\right) + 4xP + 2w\gamma - w^2 - 1\right) + \right. \\
&\qquad 8\left(e^{2g}-1\right)e^{h-3g}\sin(\alpha) + 4\cos(\alpha)\left(e^{-2g}(\gamma - w) - 2w\right) + e^{-2h}V\cos(\alpha)\Big] \\
\gamma' &= -\left(w^2-1\right)e^{3g-h}\sin(\alpha) + 4e^{g+h}\sin(\alpha) + 4e^{2g}w\cos(\alpha) - 4x\eta P'
\end{aligned}
\tag{A2}
$$

Where $'$ refers to differentiation with respect to r. The trigonometric functions are defined through the following relation:

$$
\tan(\alpha) = \frac{Ve^{3g-h} - 12e^{g+h}\left(\left(2e^{2g}-1\right)w + \gamma\right)}{6e^{2g}\left(4e^{2h} - 4Px + w^2 - 2w\gamma + 1\right) - 6e^{4g}\left(w^2-1\right) - 8e^{2h}}
\tag{A3}
$$

And the functions V and η are defined as:

$$
\begin{aligned}
\eta &= \tfrac{e^{g}(C+w) + 2e^{h}\tan(\alpha)}{-4e^{2h-g} + e^{g}(w^2-1) + 4e^{h}w\tan(\alpha)} \\
V &= (1-w^2)(w-3\gamma) - 4(1-3xP)w + 8(\kappa + \tfrac{3x}{2}C)
\end{aligned}
\tag{A4}
$$

In the main part of this work we have set:

$$
C = 1 \tag{A5}
$$

As it seems that this is required to have a quantised Chern-Simons term when P depends on r and we also choose:

$$
\kappa = \frac{1}{2} \tag{A6}
$$

To avoid a curvature singularity in the IR.

It is possible to show that if we can solve the BPS equations at $\beta = 0$ then this solution will automatically solve the more complicated BPS of the system described in section 2.1 where β is arbitrary (see [1] for details).

Appendix B: The General Asymptotic UV Solution to the BPS Equation

In the UV, the general series solution to the BPS system of Appendix A with $C = 1$ and $\kappa = \frac{1}{2}$ is given by:

$$
\begin{aligned}
e^{2g} \quad &= c\left(e^{2r/3}\right)^2 - 1 - \frac{2c_h}{3e^{2r/3}} + \frac{33}{4c\left(e^{2r/3}\right)^2} + \frac{\frac{27c_h}{5c} - \frac{8c_w}{15}}{\left(e^{2r/3}\right)^3} + \frac{\frac{8cc_h^2+49cx-396}{9c^2} - \frac{cc_w^2}{24}}{\left(e^{2r/3}\right)^4} + \\
&\quad \frac{3c_h(20cx-537)+160cc_w}{42c^2\left(e^{2r/3}\right)^5} + \\
&\quad \frac{15c\left(864cc_hc_w-6048c_h^2+c\left(285cc_w^2-56c_\gamma\right)\right)+4(1120r(3cx-8)+3cx(495cx-32734)+465192)}{7200c^3\left(e^{2r/3}\right)^6} + \\
&\quad \frac{c_h\left(765c^3c_w^2-105440cx+943758\right)-4480cc_h^3-16cc_w(635cx+1309)}{3240c^3\left(e^{2r/3}\right)^7} + \ldots \\
e^{2h} \quad &= \frac{3}{4}c\left(e^{2r/3}\right)^2 + \frac{9}{4} + \frac{c_h}{e^{2r/3}} + \frac{3x-\frac{77}{16c}}{\left(e^{2r/3}\right)^2} + \frac{3\left(2c_w-\frac{9c_h}{c}\right)}{10\left(e^{2r/3}\right)^3} - \frac{3c^3c_w^2+32cc_h^2+88cx-1536}{96c^2\left(e^{2r/3}\right)^4} + \\
&\quad \frac{c_h(863-36cx)+6cc_w(21cx-79)}{84c^2\left(e^{2r/3}\right)^5} + \\
&\quad \frac{15c\left(-416cc_hc_w+672c_h^2+3c\left(8c_\gamma-75cc_w^2\right)\right)-4(480r(3cx-8)+3cx(1755cx-14006)+99728)}{3200c^3\left(e^{2r/3}\right)^6} + \\
&\quad \frac{c_h\left(-765c^3c_w^2+37280cx-206262\right)+640cc_h^3+4cc_w(5165cx-15326)}{2160c^3\left(e^{2r/3}\right)^7} + \ldots \\
w \quad &= \frac{2}{c\left(e^{2r/3}\right)^2} + \frac{c_w}{\left(e^{2r/3}\right)^3} + \frac{22-6cx}{c^2\left(e^{2r/3}\right)^4} + \frac{32c_h+27cc_w}{6c^2\left(e^{2r/3}\right)^5} + \frac{3c^2c_hc_w-16cx+51}{2c^3\left(e^{2r/3}\right)^6} + \\
&\quad \frac{c_h(436-165cx)+cc_w(45cx+7)}{30c^3\left(e^{2r/3}\right)^7} + \ldots \\
\gamma \quad &= \frac{1}{3} + \frac{x}{\left(e^{2r/3}\right)^2} + \frac{2c_w-\frac{2c_h}{c}}{\left(e^{2r/3}\right)^3} + \frac{\frac{16r(8-3cx)}{3c^2}+c_\gamma}{\left(e^{2r/3}\right)^4} + \frac{3c_h(cx-7)+cc_w(cx-11)}{c^2\left(e^{2r/3}\right)^5} + \\
&\quad \frac{c^2\left(-9c_hc_w+cc_w^2-6c_\gamma\right)+4(8r(3cx-8)-7cx+24)}{3c^3\left(e^{2r/3}\right)^6} + \\
&\quad \frac{c_h\left(-12c^2c_\gamma+64r(3cx-8)-207cx+930\right)+9cc_w(78-5cx)}{18c^3\left(e^{2r/3}\right)^7} + \ldots \\
e^{2(\phi-\phi_\infty)} \quad &= 1 + \frac{3cx+4}{c^2\left(e^{2r/3}\right)^4} + \frac{2(c_w+3)}{5c\left(e^{2r/3}\right)^5} - \frac{4(cx+2)}{c^3\left(e^{2r/3}\right)^6} + \frac{3c_w(6cx+1)-48cx-143}{21c^2\left(e^{2r/3}\right)^7} + \\
&\quad \frac{-8c^3c_w-3c\left(8c\left(c+6x^2\right)-283x\right)+1136}{24c^4\left(e^{2r/3}\right)^8} + \frac{c_w(63-68cx)+1546cx+3429}{90c^3\left(e^{2r/3}\right)^9} + \\
&\quad \frac{3c^3c_w(c_w(15cx-71)-60cx+244)-20\left(78c^3+1267\right)cx+2468c^3+5190c^2x^2-43050}{225c^5\left(e^{2r/3}\right)^{10}} + \ldots
\end{aligned}
\tag{B1}
$$

As one would expect of a general solution of 4 coupled 1st order ODEs, these series depend on 4 integration constants c, c_h, c_w, c_γ. There is also ϕ_∞ but this is not independent. In the main part of this paper the choice $c_h = c_w = 0$ was made, this sets all odd powers of

$e^{2r/3}$ in e^{2g}, e^{2h}, w and γ to zero, simplifying the expansions considerably. When numerical matching of the IR and UV asymptotic solutions was performed it was also found that $c_\gamma = 0$ in all cases considered. It seems likely that solutions with more constants turned on do not match to the IR of Eq.2.15.

Notice that the leading order of all the functions in Eq.B1 depend only on c and so we must still have a metric that is asymptotically AdS_4 for arbitrary values of (c_h, c_w, c_γ). The more relevant question is which combinations of these values lead to a regular IR. Experience of similar Conifold solutions [26, 27] suggests it should be possible find G_2 cone solutions which exhibit a similar walking type behaviour. Confirming this however, shall be left to future work.

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